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Painlevé analysis and similarity solutions of Burgers' equation with variable coefficients

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Abstract. In this paper the Painlevé test of a generalized Burgers' equation containing an arbitrary function of time that describes nonuniformity effects is performed. Moreover the Lie group techniques allowing one to characterize some classes of functional forms for the arbitrary function and the related similarity solutions are applied.

1. Introduction

In this paper, within the framework of the Painlevé analysis for partial differential equations and of the Lie group approach, we aim to point our attention to the following generalized Burgers' equation:

$$\Omega = u_t + uu_x - u_{xx} + f(t)u = 0 \tag{1.1}$$

(as usual subscripts stand for partial derivatives with respect to the indicated variables) $f(t) \in C^3(R)$ being an arbitrary function of time.

The classical Burgers' equation (which is obtained from (1.1) when $f(t) = 0$) represents the simplest model equation suitable to describe wave propagation phenomena when there is a balance between linear evolution, quadratic nonlinearity and viscous diffusion. Apart from its applicability to various physical situations [1, 2, 3, 4], the classical Burgers' equation is remarkable because of its linearization through the well known Cole–Hopf transform mapping it to the linear heat equation [5, 6]. Unfortunately there is no Bäcklund transformation (analogous to the Cole–Hopf transform) that is able to work with generalized Burgers' equations containing variable coefficients or non-homogeneity terms [7].

Nevertheless equation (1.1) is an evolution equation of relevant interest in mathematical physics: for example it arises in connection to wave propagation in homogeneous media, $f(t)$ being related to the front wave geometry (in particular it is $f(t) = 1/t$ for spherical symmetry and $f(t) = 1/2t$ for cylindrical symmetry) [4, 8].

Furthermore it must be stressed that several asymptotic methods proposed to study multidimensional wave propagation through nonlinear media [9] lead to transport equations valid along the characteristic rays of a hyperbolic associated system, which reduce to the form (1.1) when dissipation dominates. Finally it must be remarked that the equation (1.1) is strictly related to the equation

$$u_t + g(t)uu_x - h(t)u_{xx} = 0 \tag{1.2}$$

obtained in several problems (see [9] and the references therein quoted), which, by means of the variable transformation

$$v(x,\tau)=\frac{g(t)}{h(t)}\,u(x,t)\,,\qquad \tau=\int h(t)\,\mathrm{d}t \tag{1.3}$$

is reduced to

$$v_\tau+vv_x-v_{xx}+\frac{\mathrm{d}}{\mathrm{d}\tau}\ln\!\left(\frac{h(t)}{g(t)}\right)v=0\,, \tag{1.4}$$

which is just the equation (1.1).

Various generalized Burgers' equations have been analyzed in [10, 11, 12] where, through the use of a self-similar approach, the link has been shown with a class of second order ordinary differential equations defining the so-called Euler–Painlevé transcendents that play the same role as the Painlevé equations for the Korteweg–de Vries type of equations.

In the following sections we will apply the Painlevé test for partial differential equations, as defined by Weiss, Tabor and Carnevale [13, 14], in order to identify the integrable cases and we will use the Lie group techniques [15, 16, 17, 18, 19] in order to determine some classes of functional forms for the function $f(t)$ compatible with the existence of similarity solutions of equation (1.1).

2. Painlevé test

As a first step we will perform a singular point analysis for the equation (1.1) in order to establish whether the equation under consideration has the so-called Painlevé property for partial differential equations [13, 14]. Roughly speaking, a partial differential equation is said to possess the Painlevé property if its solutions are single-valued in a neighbourhood of a non-characteristic movable singularity manifold. More precisely, let

$$\Phi(z_1,z_2,\ldots,z_n)=0 \tag{2.1}$$

represent the singularity manifold and let $u(z_1,z_2,\ldots,z_n)$ be a solution of the partial differential equation. Then seek an expansion of u in the form of a Laurent series:

$$u(z_1,z_2,\ldots,z_n)=\sum_{j=0}^{\infty}\Phi^{j+\lambda}u_j\,, \tag{2.2}$$

where $\Phi=\Phi(z_1,z_2,\ldots,z_n)$ and $u_j=u_j(z_1,z_2,\ldots,z_n)$ are analytic functions of $(z_1,z_2,\ldots,z_n)$ in a neighbourhood of the manifold (2.1). The Painlevé property is satisfied if λ is an integer and the ansatz (2.2) is correct with the requisite number of arbitrary functions; as a consequence it is conjectured that the equation is integrable. Furthermore, when a partial differential equation passes the Painlevé test then it is possible to construct a Bäcklund transform by simply truncating the expansion (2.2) at the constant level term. In particular, in [13] it has been shown by means of the Painlevé test that the classical Burgers' equation possesses a Bäcklund transform that generalizes the Cole–Hopf transform.

For the equation (1.1) a leading order analysis shows that $\lambda=-1$; consequently the expansion about the singular manifold assumes the form

$$u = \sum_{j=0}^{\infty} \Phi^{j-1} u_j \,. \tag{2.3}$$

The substitution of (2.3) into (1.1) gives the following recursion relations for $u_j(x, t)$:

$$\begin{aligned} &u_{j-2,t} + (j-2)u_{j-1}\Phi_t + \sum_{m=0}^{j} u_{j-m}(u_{m-1,x} + (m-1)\Phi_x u_m) \\ &\quad - [u_{j-2,xx} + 2(j-2)u_{j-1,x}\Phi_x + (j-2)u_{j-1}\Phi_{xx} + (j-1)(j-2)u_j\Phi_x^2] \\ &\quad + f(t)u_{j-2} = 0 \,. \end{aligned} \tag{2.4}$$

It is easily seen that the relations (2.4) are not defined when $j = -1, 2$. These values are called 'resonances' and correspond to the fact that Φ and u_2 are arbitrary functions. From (2.4) we obtain

$$\begin{aligned} j &= 0\,, \qquad u_0 = -2\Phi_x \,, \\ j &= 1\,, \qquad u_1 = \frac{\Phi_{xx} - \Phi_t}{\Phi_x} \,, \\ j &= 2\,, \qquad u_{0,t} + (u_0 u_1)_x - u_{0,xx} + f(t)u_0 = 0 \,. \end{aligned} \tag{2.5}$$

Taking account of $(2.5)_1$ and $(2.5)_2$, the relation $(2.5)_3$ (which represents a compatibility condition since it involves already determined quantities) reduces to

$$f(t)\Phi_x = 0 \tag{2.6}$$

from which it follows that equation (1.1) possesses the Painlevé property if and only if $f(t) = 0$, i.e. it reduces to the classical Burgers' equation. In a certain sense this negative result is consistent with the one obtained in [7] and stating that there are no nonlinear parabolic equations, other than Burgers' equation and the linear equation, which have Bäcklund transformations either onto themselves or onto a different parabolic equation.

3. Determination of the infinitesimal Lie groups of invariants

In this section we want to use the Lie group techniques in order to characterize the functional dependence of $f(t)$ compatible with the existence of invariance groups of the equation (1.1) and the determination of possible classes of exact similarity solutions. Due to the negative results obtained in [7] and also in the previous section, the study of the invariance of the equation (1.1) under a one-parameter Lie group of transformations [20], besides its own interest, becomes in the case at hand very important if we want to get information about the functional form of possible solutions of equation (1.1).

The application of the well known algorithm [15, 16, 17, 18, 19] leads us first to consider the most general non-extended (involving the variables x, t, u only) differential operator

$$\Xi = \xi_0(x, t, u)\,\frac{\partial}{\partial t} + \xi_1(x, t, u)\,\frac{\partial}{\partial x} + \eta(x, t, u)\,\frac{\partial}{\partial u} \tag{3.1}$$

of the Lie group and then to build, in the spirit of Lie's theory, the first and the second extensions

$$\Xi_1 = \Xi + \eta_{[x]} \frac{\partial}{\partial u_x} + \eta_{[t]} \frac{\partial}{\partial u_t}, \tag{3.2}$$

$$\Xi_2 = \Xi_1 + \eta_{[xx]} \frac{\partial}{\partial u_{xx}} + \eta_{[tt]} \frac{\partial}{\partial u_{tt}} + \eta_{[xt]} \frac{\partial}{\partial u_{xt}}. \tag{3.3}$$

The requirement that the equation (1.1) is invariant with respect to the group whose infinitesimal generators are ξ_0, ξ_1, η is equivalent to the relation

$$\Xi_2 \Omega = 0 \tag{3.4}$$

from which, taking account of (1.1) to eliminate u_t, the following invariance condition arises:

$$\begin{aligned}
&[\eta_t - f(t)u\eta_u + f(t)u\xi_{0t} - f^2(t)u^2\xi_{0u} + u\eta_x \\
&\quad + f(t)u^2\xi_{0x} - \eta_{xx} - f(t)u\xi_{0xx} + \xi_0 f'(t)u + \eta f(t)] \\
&\quad + [u\xi_{0t} - \xi_{1t} - f(t)u^2\xi_{0u} + f(t)u\xi_{1u} + \eta + u^2\xi_{0x} \\
&\quad - u\xi_{1x} - 2\eta_{xu} + \xi_{1xx} - u\xi_{0xx} - 2\xi_{0xu}f(t)u]u_x \\
&\quad + [2\xi_{1x} - \xi_{0t} + f(t)u\xi_{0u} - u\xi_{0x} + \xi_{0xx}]u_{xx} + [2\xi_{0xu} + 2\xi_{1u}]u_x u_{xx} \\
&\quad + [-\eta_{uu} + 2\xi_{1xu} - 2u\xi_{0xu} - f(t)u\xi_{0uu}]u_x^2 + [\xi_{1uu} - u\xi_{0uu}]u_x^3 + [\xi_{0uu}]u_x^2 u_{xx} \\
&\quad + [2\xi_{0x}]u_{xt} + [2\xi_{0u}]u_x u_{xt} = 0 .
\end{aligned} \tag{3.5}$$

The derivatives u_x, u_{xx}, ... must be considered independent variables, so the condition (3.5) is identically satisfied if all the coefficients of the derivatives of u vanish. The integration of the resulting determining equations leads, after some calculations, to the following result:

$$\begin{aligned}
&\xi_0(t) = 2\int A(t)\,\mathrm{d}t , \\
&\xi_1(x, t) = A(t)x + B(t) , \\
&\eta(x, t, u) = -A(t)u + A'(t)x + B'(t) ,
\end{aligned} \tag{3.6}$$

where $A(t)$ and $B(t)$ represent two functions of time t subjected, together with the function $f(t)$, to the following compatibility condition:

$$[2f(t)\int A(t)\,\mathrm{d}t]'u + [A''(t) + A'(t)f(t)]x + [B''(t) + B'(t)f(t)] = 0 \tag{3.7}$$

(here and in the sequel the prime $'$ will always denote differentiation with respect to the argument).

If $f(t) = 0$ then (3.6) and (3.7) obviously give the infinitesimals of the invariance group of the classical Burgers' equation [18, 21]. If $f(t) \neq 0$ then after simple algebra we get:

$$\xi_0(t) = \frac{2\alpha}{f(t)}\,,$$
$$\xi_1(x,t) = \left(\frac{\alpha}{f(t)}\right)' x + \beta \int \left[\exp\left(-\int f(t)\,\mathrm{d}t\right)\right] \mathrm{d}t + \gamma\,, \tag{3.8}$$
$$\eta(x,t,u) = -\left(\frac{\alpha}{f(t)}\right)' u + \left(\frac{\alpha}{f(t)}\right)'' x + \beta \exp\left(-\int f(t)\,\mathrm{d}t\right)$$

(α, β, γ are arbitrary constants) provided the function $f(t)$ is a solution of the following nonlinear third order ordinary differential equation:

$$\left(\frac{1}{f(t)}\right)''' + f(t)\left(\frac{1}{f(t)}\right)'' = 0\,. \tag{3.9}$$

On the contrary, if $f(t)$ does not satisfy equation (3.9) then we get, reconsidering (3.6) and (3.7), the so-called universal group of equation (1.1), i.e. the invariance group corresponding to $f(t)$ arbitrarily given:

$$\xi_0 = 0\,,$$
$$\xi_1(t) = \beta \int \left[\exp\left(-\int f(t)\,\mathrm{d}t\right)\right] \mathrm{d}t + \gamma\,, \tag{3.10}$$
$$\eta(t) = \beta \exp\left(-\int f(t)\,\mathrm{d}t\right).$$

4. Classes of similarity solutions

Once the infinitesimals of the group leaving equation (1.1) invariant have been determined, the similarity solutions are obtained by solving the so-called invariant surface condition:

$$\xi_0 u_t + \xi_1 u_x = \eta\,. \tag{4.1}$$

When equation (3.9) is not fulfilled, then the integration of the invariant surface condition leads, through the use of equation (1.1), to:

$$u = \frac{\exp\left(-\int f(t)\,\mathrm{d}t\right)}{\beta \int \left[\exp\left(-\int f(t)\,\mathrm{d}t\right)\right] \mathrm{d}t + \gamma} (\beta x + K_0) \tag{4.2}$$

(K_0 is an integration constant) representing the most general similarity solution of (1.1) compatible with an arbitrary $f(t)$. However, this solution is not so useful because it is linear in x and does not take into account the dissipative effects related to the occurrence in (1.1) of the second spatial derivative.

Thus, if we want to construct less trivial solutions, we have to consider equation (3.9) which provides some information about the functional form of $f(t)$ allowing for the existence of invariant solutions. By simple inspection it is possible to see that (3.9) admits the solutions:

$$f = K\,, \qquad f(t) = \frac{K}{t + t_0}\,, \tag{4.3}$$

where K and t_0 represent arbitrary constants (nevertheless in the sequel we shall limit ourselves to take, without loss of generality, $t_0 = 0$).

In any case we note here, for the sake of completeness, that it is possible to perform a systematic integration of equation (3.9), but the resulting solutions for $f(t)$ unfortunately appear only in implicit or in parametric form.

If $f = K$ then the infinitesimal generators of the group become:

$$\begin{aligned} \xi_0 &= \frac{2\alpha}{K}\,, \\ \xi_1(t) &= -\frac{\beta}{K}\exp(-Kt) + \gamma\,, \\ \eta(t) &= \beta\exp(-Kt)\,. \end{aligned} \tag{4.4}$$

It is easily seen that the invariant solutions obtained by setting $\beta = 0$ are the travelling wave solutions of (1.1). On the contrary if $\beta \neq 0$ then we get:

$$u = -\frac{\beta}{2\alpha}\exp(-Kt) + \psi(\omega) \tag{4.5}$$

with the similarity variable given by

$$\omega = \frac{2\alpha}{K}x - \gamma t - \frac{\beta}{K^2}\exp(-Kt) \tag{4.6}$$

and the similarity function $\psi(\omega)$ a solution of

$$-\gamma\psi' + \frac{2\alpha}{K}\psi\psi' - \frac{4\alpha^2}{K^2}\psi'' + K\psi = 0\,, \tag{4.7}$$

which is the same ordinary differential equation as that arising when we try for a travelling wave solution. It seems interesting to notice that if $K > 0$ then the solution (4.5) approaches asymptotically to a travelling wave solution because either $\omega \to (2\alpha x/K - \gamma t)$ or $u \to \psi(\omega)$ when $t \to \infty$. Moreover, it is worth noting that equation (4.7) can be, at least in principle, completely solved by means of a first order differential equation and a quadrature (the resulting solution will be given in parametric form) essentially because it does not contain explicitly the independent variable ω.

By taking $f(t) = K/t$ we have to distinguish among the three following cases:

$$\text{i) } K \neq \frac{1}{2}, 1\,, \qquad \text{ii) } K = \frac{1}{2}\,, \qquad \text{iii) } K = 1\,.$$

In the cases i), ii) the infinitesimal generators of the Lie group are given by:

$$\xi_0(t) = \frac{2\alpha}{K}t\,,$$

$$\xi_1(x, t) = \frac{\alpha}{K} x + \frac{\beta}{1-K} t^{1-K} + \gamma ,$$
$$\eta(t, u) = -\frac{\alpha}{K} u + \beta t^{-K} , \tag{4.8}$$

whereas in the latter case we get:

$$\xi_0(t) = 2\alpha t ,$$
$$\xi_1(x, t) = \alpha x + \beta \ln t + \gamma , \tag{4.9}$$
$$\eta(t, u) = -\alpha u + \frac{\beta}{t} .$$

For the spherical Burgers' equation ($K = 1$) we find the following similarity solution:

$$u = -\frac{\beta}{\alpha t} + t^{-1/2}\psi(\omega) \tag{4.10}$$

with the similarity variable

$$\omega = \left(x + \frac{\beta}{\alpha} \ln t + \frac{2\beta + \gamma}{\alpha}\right) t^{-1/2} \tag{4.11}$$

and the function $\psi(\omega)$ a solution of the differential equation:

$$\frac{\psi}{2} - \frac{\omega\psi'}{2} + \psi\psi' - \psi'' = 0 . \tag{4.12}$$

On the contrary for the cylindrical Burgers' equation ($K = 1/2$) it is easy to obtain the solution

$$u = \frac{\beta}{4\alpha} t^{-1/2} \ln t + t^{-1/2}\psi(\omega) , \tag{4.13}$$

where

$$\omega = \left(x + \frac{\gamma}{2\alpha}\right) t^{-1/2} - \frac{\beta}{2\alpha} \ln t \tag{4.14}$$

and $\psi(\omega)$ such that

$$-\frac{\beta}{2\alpha} \psi' - \frac{\omega\psi'}{2} + \psi\psi' - \psi'' + \frac{\beta}{4\alpha} = 0 . \tag{4.15}$$

Finally, for $K \neq 1/2$ and $K \neq 1$, the integration of the invariant surface condition leads to:

$$u = \frac{\beta}{\alpha} \frac{K}{1-2K} t^{-K} + t^{-1/2}\psi(\omega) \tag{4.16}$$

with

$$\omega = \left(x - \frac{\beta}{\alpha} \frac{K}{(1-K)(1-2K)} t^{1-K} + \frac{\gamma K}{\alpha}\right) t^{-1/2} \tag{4.17}$$

and the similarity reduced equation having the form

$$\left(K-\frac{1}{2}\right)\psi-\frac{\omega\psi'}{2}+\psi\psi'-\psi''=0\,. \tag{4.18}$$

A simple inspection of equation (4.15) shows the existence of the trivial solution

$$\psi(\omega)=\frac{\omega}{2} \tag{4.19}$$

that obviously does not furnish meaningful solutions of the cylindrical Burgers' equation; however, it is possible to use the solution (4.19) to rewrite equation (4.15) more simply. In fact, making the substitution

$$\psi(\omega)=\phi(\omega)+\frac{\omega}{2} \tag{4.20}$$

the equation (4.15) transforms to the following autonomous form:

$$\frac{\phi}{2}+\phi\phi'-\frac{\beta}{2\alpha}\phi'-\phi''=0\,. \tag{4.21}$$

Because ω does not appear explicitly, the considerations made above for the equation (4.7) can be repeated for the equation (4.21).

When the function $f(t)$ does not take any of the forms (4.3) but is still a solution of the equation (3.9), it is also possible to determine the expression of the invariant solutions and the form of the reduced ordinary differential equation that must be satisfied by the corresponding similarity function.

In order to pursue this task we will proceed step by step. First of all we will suppose that only the group constant α is different from zero. By choosing, without loss of generality, $\alpha=1$, the infinitesimal generators of the Lie group reduce to

$$\begin{aligned}
\xi_0(t)&=\frac{2}{f(t)}\,,\\
\xi_1(x,t)&=\left(\frac{1}{f(t)}\right)'x\,,\\
\eta(x,t,u)&=-\left(\frac{1}{f(t)}\right)'u+\left(\frac{1}{f(t)}\right)''x\,,
\end{aligned} \tag{4.22}$$

with the function $f(t)$ a solution of the equation (3.9) and such that $(1/f(t))''\neq 0$ (in order to exclude the cases previously considered).

The integration of the characteristic equations corresponding to the invariant condition leads to the following result (hereafter we shall assume $f(t)$ to be a non-negative function of time):

$$\begin{aligned}
\omega&=x\sqrt{f(t)}\,,\\
u&=\psi(\omega)\sqrt{f(t)}+\frac{x}{2}\,f(t)\left(\frac{1}{f(t)}\right)'\,,
\end{aligned} \tag{4.23}$$

ω representing as usual the similarity variable and $\psi(\omega)$ the corresponding similarity function.

The insertion of $(4.23)_2$ into the equation (1.1) allows us to obtain, after some simple calculations, the following ordinary differential equation:

$$\psi\psi' - \psi'' + \psi + \frac{\omega}{2}\left\{\frac{1}{f(t)}\left(\frac{1}{f(t)}\right)'' - \frac{1}{2}\left(\frac{1}{f(t)}\right)'^2 + \left(\frac{1}{f(t)}\right)'\right\} = 0\,. \tag{4.24}$$

Bearing in mind the constraint (3.9) it is immediately seen that the terms enclosed in the curly brackets represent a constant (due to equation (3.9) their time derivative is vanishing). If we call this constant $2M$ then the ordinary differential equation that must be satisfied by the similarity function $\psi(\omega)$ results as

$$\psi\psi' - \psi'' + \psi + M\omega = 0\,. \tag{4.25}$$

In the case in which we take the group constants α and γ different from zero, the results we get can be summarized as follows:

$$\begin{aligned} \omega &= x\sqrt{f(t)} - \frac{\gamma}{2\alpha}\int f^{3/2}(t)\,\mathrm{d}t\,, \\ u &= \psi(\omega)\sqrt{f(t)} + \frac{x}{2}\, f(t)\left(\frac{1}{f(t)}\right)' + \frac{\gamma}{2\alpha}\, f(t)\,. \end{aligned} \tag{4.26}$$

As in the previous case, it is easy to ascertain that the function $\psi(\omega)$ must satisfy the equation (4.25) with the same meaning for the constant M, whereas the variable ω is given by $(4.26)_1$.

Finally, in the most general case in which all the group constants are assumed non-vanishing, the consideration of the invariant surface condition provides us the following similarity solution:

$$\begin{aligned} \omega &= x\sqrt{f(t)} - \frac{1}{2\alpha}\int f^{3/2}(t)\left(\beta\int \exp\left[-\int f(t)\,\mathrm{d}t\right] + \gamma\right)\mathrm{d}t \\ u &= \psi(\omega)\sqrt{f(t)} + \frac{x}{2}\, f(t)\left(\frac{1}{f(t)}\right)' + \frac{\beta}{2\alpha}\, f(t)\int \exp\left[-\int f(t)\,\mathrm{d}t\right]\mathrm{d}t + \frac{\gamma}{2\alpha}\, f(t)\,. \end{aligned} \tag{4.27}$$

Also in this case we are able to establish that the reduced similarity equation for the function $\psi(\omega)$ is given by equation (4.25).

It is a simple matter to recognize that equation (4.25) admits the following real solution

$$\psi(\omega) = A\omega \tag{4.28}$$

with $A =$ constant such that $A^2 + A + M = 0$ provided that $M \leqslant 1/4$.

In this case the transformation

$$\psi(\omega) = \phi(\omega) + A\omega \tag{4.29}$$

reduces equation (4.25) to the form

$$(A+1)\phi + A\omega\phi' + \phi\phi' - \phi'' = 0 \tag{4.30}$$

which exactly reproduces the similarity equation (4.12) obtained above for the spherical Burgers' equation when we choose $A = -1/2$ (corresponding to $M = 1/4$).

If we take $M = 0$, which implies that the compatibility equation (3.9) admits the following implicit solution:

$$\frac{2}{f_0^2}\left(\frac{f_0}{\sqrt{f(t)}} + 2\right) - \frac{4}{f_0^2}\ln\left(\frac{f_0}{\sqrt{f(t)}} + 2\right) = t + t_0 \tag{4.31}$$

(f_0 and t_0 representing arbitrary integration constants); then equation (4.25) appears in autonomous form; so the same considerations made for equations (4.7) and (4.15) apply also to equation (4.25).

It has to be remarked that all the reduced similarity equations (namely equations (4.7), (4.12), (4.18), (4.21) and (4.30)) to be solved in order to get the requested invariant solutions are easily transformed, by using as new dependent variable the old one raised to the power -1, to particular forms of the Euler–Painlevé equations identified in [10, 11, 12] as characterizing a wide class of generalized Burgers' equations.

Finally, we briefly notice that the application of the procedure proposed by Bluman and Cole [22] (see also [16]) in order to determine possible non-classical symmetries of the equation at hand by making use both of the given equation (1.1) and of the invariant surface condition (4.1) does not produce new results. In fact, if $f(t) \neq 0$ then the non-classical approach furnishes the same symmetry groups as those provided by the classical analysis performed in Section 3. On the contrary, if $f(t) = 0$ (i.e. in the case of the classical Burgers' equation) the method of Bluman and Cole [22] works, and it is possible to get truly non-classical symmetries [16].

6. Conclusions

In this paper we considered a generalized Burgers' equation containing an arbitrary function $f(t)$ that describes the inhomogeneity effects (or, equivalently, the time-dependence of the coefficients). First of all we have shown that there is no non-trivial case passing the Painlevé test for partial differential equations. Then we have employed the Lie group methods in order to identify the constraints to be imposed on the arbitrary function involved in the equation at hand allowing one to determine the infinitesimal Lie groups of invariants. Furthermore some classes of exact similarity solutions have been characterized and discussed in connection with special functional forms of the arbitrary function $f(t)$.

The ordinary differential equations arising from the group reduction are autonomous (equation (4.7) valid for $f(t) = \text{constant}$) or are easily transformed to autonomous form (equation (4.15) valid for $f(t) = t/2$ and equation (4.25) valid for $f(t) \neq K/t$ and satisfying the constraint (3.9)) or are immediately reduced to particular Euler–Painlevé equations (equation (4.12) valid for $f(t) = 1/t$ and equation (4.18) valid for $f(t) = K/t$ with $K \neq 1, 1/2$).

Finally, the Bluman and Cole [22] non-classical approach for the determination of invariance groups has been used and it has been shown that this procedure does not allow the introduction, for the equation at hand, of similarity reductions different from those known from the application of the classical Lie algorithm.

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